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PROPERTIES AND APPLICATIONSOF GENERALIZED MATRIX HYPERGEOMETRIC FUNCTIONS

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ABSTRACT

In this paper, we construct a (p, k)-hypergeometric function by using the Hadamard product, which we call the generalized (p, k)-hypergeometric function. Several properties, namely, convergence properties, derivative formulas, integral representations and differential equations are indicated of this function. The latter function is a generalization of the usual hypergeometric function, the k-analogue of hypergeometric function and other hypergeometric functions are recently presented. As an application, we obtain the solution of the generalized fractional kinetic equations involving of the generalized (p, k)-hypergeometric function.

keywords: Hypergeometric Functions; Generalized Matrix

INTRODUCTION

OVERTURE Recently, the roles of the hypergeometric functions in solutions to boundary value problems, and its use in various problems of applied mathematics, physics, chemistry, engineering and other branches of science are well known. The reader may consult, for instance, Refs. 1–4. Further, from hypergeometric function appearance time until now, several special functions such as Jacobi, Bessel, Laguerre, Humbert, Chebyshev polynomials and other functions have been presented and some of their applications have been investigated in the literature works. Recently, various extensions of the hypergeometric functions have been considered by several authors, which have very powerful applications (see Refs. 5–8). In particular, the extended k-analogue of hypergeometric functions had recently been defined, and some of their properties have been discussed in Refs. 9–12. motivated and connected with foregoing works. This paper makes a new contribution of the extended k-analogue of hypergeometric functions by using the Hadamard product and studies their applications. Also, it generalizes the results of the recent work of Abdalla and Hidan.13 This paper is arranged as follows. In Sec. 2, we give preliminaries and terminologies that are needed in this paper. In Sec. 3, we present construction of the (p, k)-hypergeometric function by using the Hadamard product, which is the so-called the generalized (p, k)-hypergeometric function (G(p, k)-HF). We also deduce its convergence properties. Various properties, such as differentiation formulae, integral representations, integral transforms and differential equations connecting the G(p, k)-HF, are reported in Sec. 4. As an application, we establish the solutions of generalized kinetic equation involving G(p, k)-HF with the help of Pσ-transform

PRELIMINARIES

Throughout this paper, N := {1, 2, 3,...} denotes the set of natural numbers, N0 = N ∪ {0}, Z− := {−1, −2, −3,...} denotes the set of negative integers, Z− 0 = Z− ∪ {0}, R+ denotes the set of positive real numbers and C denotes the set of complex numbers.

In Ref. 5, Diaz and Pariguan introduced the khypergeometric function as follows:

$$\mathcal{U} = {}_2\mathbf{F}_1^k \left[\begin{matrix} (\beta_1; k), (\beta_2; k) \\ (\beta_3; k) \end{matrix} ; u \right] = \sum_{m=0}^{\infty} \frac{(\beta_1)_{m,k} (\beta_2)_{m,k}}{(\beta_3)_{m,k}} \cdot \frac{u^m}{m!}, \quad |u| < \frac{1}{k},$$

where (θ)m,k is the k-Pochhammer symbol given by

$$(\theta)_{m,k} = \frac{\Gamma^k(\theta + mk)}{\Gamma^k(\theta)} = \begin{cases} \theta(\theta + k) \dots (\theta + (m - 1)k), & m \in \mathbb{N}, \theta \in \mathbb{C} \\ 1, & m = 0, \ k \in \mathbb{R}^+, \ \theta \in \mathbb{C} \setminus \{0\}, \end{cases}$$

and Γk(θ) is the k-gamma function defined by

$$\Gamma^k(\theta) = \int_0^\infty x^{\theta - 1} e^{-\frac{x^k}{k}} dx = \lim_{n \to \infty} \frac{n! k^n (nk)^{\frac{\theta}{k} - 1}}{(\theta)_{n,k}}, \quad \theta \in \mathbb{C} \setminus kZ^-.$$

For k → 1 in (1), we get the Gauss hypergeometric function

$${}_2\mathbf{F}_1 \left[\begin{matrix} \beta_1, \beta_2 \\ \beta_3 \end{matrix} ; u \right] = \sum_{m=0}^{\infty} \frac{(\beta_1)_m (\beta_2)_m}{(\beta_3)_m} . \frac{u^m}{m!}, \quad |u| < 1,$$

where β1, β2, β3 are complex parameters with β3 ∈ C\Z− 0 , and

$$(\beta_1)_m = \frac{\Gamma(\beta_1 + m)}{\Gamma(\beta_1)} = \begin{cases} \beta_1(\beta_1+1)\dots(\beta_1 \\ \quad +m-1), & m \in \mathbb{N},\ \beta_1 \in \mathbb{C}, \\ 1, & m = 0;\ \beta_1 \in \mathbb{C}\backslash \end{cases}$$

is the usual Pochhammer's symbol and Γ(.) is the classical gamma function (see, e.g. Refs. 3 and 4). Further, the k-beta function Bk(s, t) is defined in Ref. 5 by

$$\mathcal{B}^k(t,s) = \begin{cases} \frac{1}{k}\int_0^1 x^{\frac{t}{k}-1}(1 & (k \in \mathbb{R}^+, \min\{\mathrm{Re}(t), \\ \quad -x)^{\frac{s}{k}-1}dx, & \mathrm{Re}(s)\} > 0), \\ \frac{\Gamma^k(t)\Gamma^k(s)}{\Gamma^k(t+s)}, & (k \in \mathbb{R}^+, t, s \in \mathbb{C}\backslash\mathbb{Z}_0^-). \end{cases}$$

For k = 1 in (6) reduces to the classical beta function

$$\mathbf{B}(t,s) = \int_0^1 x^{t-1}(1-x)^{s-1}dx, \quad \mathrm{Re}(t) > 0\ \ \mathrm{Re}(s) > 0.$$

The relation between the k-beta function Bk(t, s) and the classical beta function B(t, s) is

$$\mathcal{B}^k(t,s) = \frac{1}{k}\mathbf{B}\left(\frac{t}{k}, \frac{s}{k}\right)$$

The k-hypergeometric function in (1) satisfies the following differential equation

$$ku(1-ku)\mathcal{U}'' + [\beta_3 - (\beta_1 + \beta_2 + k)ku] \times \mathcal{U}' - \beta_1\beta_2\mathcal{U} = 0.$$

In Refs. 9–11, authors defined the (p, k)- analogues of hypergeometric function

$$\mathcal{W}(p,k;w) = {}_2\mathbf{H}_1^{(p,k)}\left[\begin{matrix}(\alpha_1;k),(\alpha_2;k) \\ (\alpha_3;k)\end{matrix};w\right] = \sum_{r=0}^{\infty}\frac{(\alpha_1)_{r,k}(\alpha_2)_{r,k}}{(\alpha_3)_{r,k}}.\frac{w^r}{(rp)!},$$

which is an entire function for p > 1, where k ∈ R+ and α1, α2, w ∈ C and α3 ∈ C \ Z− 0 , and (α)j,k is the k-Pochhammer symbol defined in (2). They also obtained the following differential equation:

$$\left[\vartheta\left(\vartheta-\frac{1}{p}\right)\left(\vartheta-\frac{2}{p}\right)\cdots\left(\vartheta-\frac{p-1}{p}\right)\right.$$
$$\times(k\vartheta+\alpha_3-k)$$
$$\left.-\left\{\frac{w}{p^p}(\alpha_1+mk)(\alpha_2+mk)\right\}\right]$$
$$\times\mathcal{W}(p,k;w)=0,$$

where ϑ = w d dw is differential operator.

Some special cases of the W(p, k; w) are the type and (4). Further, at k = 1, we obtain the p-extend hypergeometric functions in the following form :

$${}_2\mathbf{H}_1^{(p)}\left[\begin{matrix}\alpha_1,\alpha_2\\ \alpha_3\end{matrix};w\right]=\sum_{r=0}^{\infty}\frac{(\alpha_1)_r\,(\alpha_2)_r}{(\alpha_3)_r}.\frac{w^r}{(rp)!},$$
$$p\in\mathbb{N},$$

which is also an entire function for p > 1.

Readers can find some properties and applications of this function in Refs. 9–11. The following concept of the Hadamard product is very useful in previous investigations,13–15 and also in our study

$$(f_1*f_2)(\xi)=(f_2*f_1)(\xi)=\sum_{n=0}^{\infty}$$
$$\times\,\alpha_n\beta_n\xi^n\quad(|\xi|<R,R\geq R_{f_1}\cdot R_{f_2}),$$

where f1(ξ) and f2(ξ) are analytic at ξ = 0. Maclaurin series of these functions with their respective radii of convergence Rf1 and Rf2 are

$$f_1(\xi)=\sum_{n=0}^{\infty}\alpha_n\xi^n\quad(|\xi|<R_{f_1}),$$

$$f_2(\xi)=\sum_{n=0}^{\infty}\beta_n\xi^n\quad(|\xi|<R_{f_2}).$$

In Refs. 16 and 17, authors have introduced the Pσ-transform (also designated as Pathway transform) as

$$\mathbf{P}_\sigma\{f(x),s\} = F(s)$$
$$= \int_0^\infty [1+(\sigma-1)s]^{\frac{-x}{\sigma-1}} f(x)d(x), \quad \sigma > 1,$$
$$\lim_{\sigma\to 1^+} [1+(\sigma-1)s]^{\frac{-x}{\sigma-1}} = e^{-sx}.$$

This transform is the generalization of the classical Laplace transform

$$\lim_{\sigma\to 1} \mathbf{P}_\sigma\{f(x),s\} = L\{f(x),s\}.$$

Applying this transform, we find that

$$\mathbf{P}_\sigma\{1,s\} = \frac{\sigma-1}{\ln[1+(\sigma-1)s]}, \quad \sigma > 1,$$
$$\mathbf{P}_\sigma\left\{\frac{x^n}{n!},s\right\} = \left\{\frac{\sigma}{\ln[1+(\sigma-1)s]}\right\}^{n+1}, \quad n \in$$

$$\mathbf{P}_\sigma\{x^{\varepsilon-1},s\} = \left\{\frac{\sigma-1}{\ln[1+(\sigma-1)s]}\right\}^{\varepsilon}\Gamma(\varepsilon),$$
$$\sigma > 1, \varepsilon \in \mathbb{C}, \mathrm{Re}(\varepsilon) > 0,$$
$$\mathbf{P}_\sigma\{{}_0\mathbb{D}_x^{-\nu} f(x),s\}$$
$$= \left[\frac{\sigma-1}{\ln[1+(\sigma-1)s]}\right]^{\lambda} \mathbf{P}_\sigma\{f(x),s\}$$
$$\times \lambda \in \mathbb{C},$$

where 0D−ν τ is the familiar Riemann–Liouville fractional integral operator (see Refs. 18 and 19) given by

$${}_0\mathbb{D}_\tau^{-\nu} f(\tau)$$
$$= \frac{1}{\Gamma(\nu)} \int_0^\tau (\tau-s)^{\nu-1} f(s)ds, \quad \nu \in \mathbb{R}^+.$$

Recently, the Pσ-transform has been applied to solve the generalized fractional kinetic equations involving the extended k-gamma function and the extended Wright (ζ-Gauss) hypergeometric function in Refs. 19–27.

DEFINITION AND CONVERGENCE PROPERTY

First, we define a generalized (p, k)-hypergeometric functions as follows.

$$ {}_2\mathbf{H}_1^{(p_\ell;k,\ell)}\begin{bmatrix} a_\ell, b_\ell \\ c_\ell \end{bmatrix}; \xi \Big] = \sum_{n=0}^{\infty} \frac{(a_\ell)_{n,k}(b_\ell)_{n,k}}{(c_\ell)_{n,k}} \frac{\xi^n}{(p_\ell n)!}; \quad \ell = 1, 2, $$

are two (p, k)-hypergeometric functions given in (9) and according to the concept of the Hadamard product (12), we have

$$ \begin{aligned} \Upsilon(p_\ell, k; \xi) &= \Upsilon^{p_1,p_2}\left[\begin{matrix}(a_1,k),(a_2,k),(b_1,k),(b_2,k)\\(c_1,k),(c_2,k)\end{matrix};\xi\right] \\ &= \left({}_2\mathbf{H}_1^{(p_1;k,1)} * {}_2\mathbf{H}_1^{(p_2;k,2)}\right)(\xi) \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_{n,k}(b_1)_{n,k}(a_2)_{n,k}(b_2)_{n,k}}{(c_1)_{n,k}(c_2)_{n,k}(np_1)!(np_2)!}\xi^n, \quad (21) \end{aligned} $$

where k ∈ R+, p ∈ N, a, b, ξ ∈ C and c ∈ C \ Z− 0 , (where = 1, 2). We shall call this function as the generalized (p, k)-hypergeometric function or briefly G(p, k)-HF.

The G(p, k)-HF is a generalization of the Hadamard product of k-Gauss hypergeometric functions, which is recent introduced in Ref. 13. For p1 = p2 = 1, the G(p, k)-HF (21) reduces to Ref. 13 of Definition 4.

Second, we show the convergence property of the power series (21).

Theorem 4. If p1 > 1, p2 > 1 and k ∈ R+, then G(p, k)-HF is an entire function of ξ.

Proof. For this proof, we can rewrite (21) in the following form:

$$ \Upsilon^{p_1,p_2}\left[\begin{matrix}(a_1,k),(a_2,k),(b_1,k),(b_2,k)\\(c_1,k),(c_2,k)\end{matrix};\xi\right] = \sum_{n=0}^{\infty} \mathbf{U}_n \xi^n, $$

Where

$$ \mathbf{U}_n = \frac{(a_1)_{n,k}(b_1)_{n,k}(a_2)_{n,k}(b_2)_{n,k}}{(c_1)_{n,k}(c_2)_{n,k}(np_1)!(np_2)!}. $$

Using the identity (θ)n+1,k = (θ+nk)(θ)n,k we compute the radius of convergence R of the above series as

$$\frac{1}{\mathrm{R}} = \lim_{n\to\infty}\left|\frac{\mathbf{U}_{n+1}}{\mathbf{U}_n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\begin{matrix}(a_1)_{n+1,k}(b_1)_{n+1,k}(a_2)_{n+1,k}\\(b_2)_{n+1,k}(np_1)!(np_2)!(c_1)_{n,k}(c_2)_{n,k}\end{matrix}}{\begin{matrix}(np_1+p_1)!(np_2+p_2)!(c_1)_{n+1,k}(c_2)_{n+1,k}\\(a_1)_{n,k}(b_1)_{n,k}(a_2)_{n,k}(b_2)_{n,k}\end{matrix}}\right|$$

$$= \lim_{n\to\infty}\frac{\begin{matrix}n^4\left(\frac{a_1}{n}+k\right)\left(\frac{b_1}{n}+k\right)\\\left(\frac{a_2}{n}+k\right)\left(\frac{b_2}{n}+k\right)\end{matrix}}{\begin{matrix}n^{p_1}\left(p_1+\frac{p_1}{n}\right)\left(p_1+\frac{p_1-1}{n}\right)\cdots\\\left(p_1+\frac{1}{n}\right)n^{p_2}\left(p_2+\frac{p_2}{n}\right)\\\left(p_2+\frac{p_2-1}{n}\right)\cdots\left(p_2+\frac{1}{n}\right)n^2\\\left(\frac{c_1}{n}+k\right)\left(\frac{c_2}{n}+k\right)\end{matrix}}$$

$$= \lim_{n\to\infty}\frac{k^2n^4}{p_1^{p_1}p_2^{p_2}\ n^{p_1+p_2+2}} = 0,$$

provide that p1 + p2 > 2, we thus have R = ∞ and hence the G(p, k)-HF (22) is entire.

For p1 = p2 = 1 and k ∈ R+, the G(p, k)-HF defined in (22)

- Converges absolutely for |ξ| < 1 k2 ;
- Converges absolutely for |ξ| = 1 under the condition

$$\frac{k}{2}\,\mathrm{Re}\left(\frac{c_1}{k}+\frac{c_2}{k}+1-\frac{a_1}{k}-\frac{a_2}{k}-\frac{b_1}{k}-\frac{b_2}{k}\right) > 0;$$

- Diverges for |ξ| > 1 k2 . For k = 1, then the G(p, k)-HF defined in (22) is an entire function of ξ.

For p1 = p2 = 1 and k = 1 in Corollary 5, we obtain the convergence property of the generalized hypergeometric function (cf. Refs. 3 and 4).

SOME PROPERTIES OF THE G(P, K)-HF

In this section, we present various properties of the G(p, k)-HF as follows:

Differentiation Formulas Theorem

The following derivative formulas hold true:

$$\frac{d^n}{d\xi^n}\left\{\Upsilon^{p_1,p_2}\left[\begin{matrix}(a_1,k),(a_2,k),(b_1,k),(b_2,k)\\(c_1,k),(c_2,k)\end{matrix};\xi\right]\right\}$$
$$=\frac{(a_1)_{n,k}(a_2)_{n,k}\,(b_1)_{n,k}(b_2)_{n,k}}{p_1^{p_1}\;p_2^{p_2}\,(c_1)_{n,k}(c_2)_{n,k}}\Upsilon^{p_1,p_2}$$
$$\times\left[\begin{matrix}(a_1+k,k),(a_2+k,k),(b_1+k,k),\\(b_2+k,k),(c_1+k,k),(c_2+k,k)\end{matrix};\xi\right],$$

$(k\in\mathbb{R}^+,p_1,p_2\in\mathbb{N},\mathrm{Re}(c_1)>0,\mathrm{Re}(c_2)>0,n\in\mathbb{N}_0)$.

$$\frac{d^n}{d\xi^n}\left\{\xi^{\frac{c_1}{k}}\Upsilon^{p_1,p_2}\left[\begin{matrix}(a_1,k),(a_2,k),(b_1,k),(b_2,k)\\(c_1+k,k),(c_2,k)\end{matrix};\xi\right]\right\}$$
$$=\frac{\xi^{\frac{c_1}{k}-n}\;\Gamma^k(c_1)}{k^{n-1}\Gamma^k(c_1-(n-1)k)}\Upsilon^{p_1,p_2}$$

$$\times\left[\begin{matrix}(a_1,k),(a_2,k),(b_1,k),(b_2,k)\\(c_1-(n-1)k,k),(c_2,k)\end{matrix};\xi\right]$$

$(k\in\mathbb{R}^+,p_1,p_2\in\mathbb{N},\ \mathrm{Re}(a_1)>0,\mathrm{Re}(a_2)>0,$ $\mathrm{Re}(b_1)>0,\mathrm{Re}(b_2)>0,\ \mathrm{Re}(c_1)>0,\mathrm{Re}(c_2)>0,$ $n\in\mathbb{N}_0)$ *and*

$$\frac{d^n}{d\xi^n}\left\{\xi^{\frac{c_2}{k}}\Upsilon^{p_1,p_2}\left[\begin{matrix}(a_1,k),(a_2,k),(b_1,k),(b_2,k)\\(c_1,k),(c_2+k,k)\end{matrix};\xi\right]\right\}$$
$$=\frac{\xi^{\frac{c_2}{k}-n}\Gamma^k(c_2)}{k^{n-1}\Gamma^k(c_2-(n-1)k)}\Upsilon^{p_1,p_2}$$

$$\times\left[\begin{matrix}(a_1,k),(a_2,k),(b_1,k),(b_2,k)\\(c_1,k),(c_2-(n-1)k,k)\end{matrix};\xi\right]$$

(k ∈ R+, p1, p2 ∈ N, Re(a1) > 0, Re(a2) > 0, Re(b1) > 0, Re(b2) > 0, Re(c1) > 0, Re(c2) > 0, n ∈ N0).

When k = 1 and p1 in (23), then we can easily obtain the generalized results in Ref. 9 of Theorem 7. Also, in the case of p1 = p2 = 1, then (24), and (25) reduce to the corresponding results

OBJECTIVE OF THE STUDY

1. To the study on Preliminaries
2. To the study on Convergence Property

CONCLUSION

We have been unable to prove the symmetry of the polynomials A, for general partition h by appealing directly to properties of the generalized hypergeometric coefficients (+Fl(a, b; c)lp). The basic difficulty is that we thus far have obtained identities between these coefficients by only using the Euler identity for the underlying generalized hypergeometric functions. This approach always leads to relations in which the denominator parameters c, c', c", . . . occurring in the sequence of (,Pi(. * -)I -) factors is the same on both sides of an equality. The structure of relation is not of this form, as seen from It appears that a proof of along these lines requires additional properties of the generalized hypergeometric series going beyond the Euler identity. We will prove in a subsequent paper by very different methods. The principal contribution of this paper has been to give the explicit properties of the generalized hypergeometric coefficients that are required to prove the desired symmetries of the SU(3) denominator function Gl and to give their integral representation as well as that of the associated function. A by-product of these investigations has been the discovery of some elegant properties of the iFs(a; z) generalized hypergeometric functions and their relation to some classic results in q-series.

REFRENCES

1. V. Akhmedova and E. Akhmedov, Selected Special Functions for Fundamental Physics, SpringerBriefs in Physics (Springer, Cham, Switzerland, 2019).
2. M. Abdalla, Special matrix functions: Characteristics, achievements and future directions, Linear Multilinear Algebra 68 (2020) 1–28.
3. I. Sneddon, Special Functions of Mathematical Physics and Chemistry (Oliver and Boyd, 1956).
4. E. Rainville, Special Functions (The Macmillan, New York, USA, 1960).
5. R. Diaz and E. Pariguan, On hypergeometric functions and k-Pochhammer symbol, Divulg. Mat. 15 (2007) 179–192.
6. F. He, A. Bakhet, M. Hidan and M. Abdalla, On the extended hypergeometric matrix functions and their applications for the derivatives of the extended Jacobi matrix polynomial, Math. Eng. 2020 (2020) 4268361.
7. R. Jana, B. Maheshwari and A. Shukla, Some results on the extended hypergeometric function, J. Indian Math. Soc. 87 (2020) 70.
8. M. Abdalla, S. A. Idris and I. Mekawy, Some results on the extended hypergeometric matrix functions and related functions, J. Math. 2021 (2021) 2046726.
9. M. Abdalla, M. Hidan, S. Boulaaras and B. Cherif, Investigation of extended k-hypergeometric functions and associated fractional integrals, Math. Probl. Eng. 2021 (2021) 9924265.
10. M. Hidan, S. Boulaaras, B. Cherif and M. Abdalla, Further results on the (p; k)-analogue of hypergeometric functions associated with fractional calculus operators, Math. Probl. Eng. 2021 (2021) 5535962.

11. M. Abdalla, S. Boulaaras, M. Akel, S. A. Idris and S. Jain, Certain fractional formulas of the extended k-hypergeometric functions, Adv. Difference Equations 2021 (2021) 450.

12. M. Chanda, J. C. Prajapatib and E. Bonyahc, Fractional integrals and solution of fractional kinetic equations involving k-Mittag-Leffler function, Trans. A. Razmadze Math. Inst. 171 (2017) 144–166.